Some Exact Analytical Solutions of Planetary Entry

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Approximate analytical solutions of entry into a planetary atmosphere at constant lift-drag ratio have been presented by Gazley, Chapman, Allen and Eggers, and Loh. Constant lift-drag ratio entry dictates variable angles of inclination. On the other hand, variable lift-drag ratio entry has been discussed by Lees, Grant, and Loh. A few approximate analytical solutions of variable lift-drag ratio entry have been given by Loh. exact analytical solutions of entry into a planetary atmosphere have not been presented before. It is the purpose of this paper to present some exact analytical solutions obtained recently. When minor terms of the exact solutions are neglected, the exact solutions presented here reduce precisely to those approximate solutions presented previously by Loh for variable lift-drag ratio entry.

Nomenclature

A = reference area for lift and drag expressions, ft^2
C = dimensional constant in heat transfer equations
C_D = drag coefficient
C_F' = equivalent skin-friction coefficient $C_F' = \frac{1}{S}\int_S C_{Fl}\left(\frac{\rho_l}{\rho}\right)\left(\frac{V_l}{V}\right) P_r^{1/2}\left(\frac{C_{pl}}{C_p}\right) dS$
C_F = skin-friction coefficient
C_L = lift coefficient
C_p = specific heat at constant pressure
D = drag, lb
g = acceleration due to force of gravity, ft/sec^2
h = convective heat transfer coefficient, ft-lb/ft^2-sec-°R
H = convective heat transferred per unit area, ft-lb/ft^2
K = constant $= 6.8$ to 15×10^{-6} $(\mu/\mu\ \text{earth})^{1/2} \times (Pr/Pr_{\text{earth}})^{-2/3} \times [(\bar{\gamma}-1)/\bar{\gamma}]^{1/4}/[(\bar{\gamma}-1)/\bar{\gamma}]^{1/4}$ earth (given by Allen and Eggers[2, 4] and Chapman[3])
k = thermal conductivity
m = mass of the vehicle, slugs
R = distance of vehicle measured from center of planet, ft. When altitude of the vehicle is small in comparison with planet radius, R may be taken approximately as R_0; R also stands for range
R_0 = radius of planet
r_c = radius of curvature of flight path (see Fig. 1), ft
Q = convective heat transferred, ft-lb
s = distance along flight path, ft
S = surface area, ft^2
t = time, sec
T = temperature, °R
V = velocity, fps
Pr = Prandtl number
ρ = atmospheric density, slugs/ft^3
μ = coefficient of viscosity, lb-sec/ft^2
θ = angle of inclination or angle of flight path to local planet horizontal, positive for descent, deg
ϵ = surface emissivity
k = Stefan-Boltzmann constant $= 3.7 \times 10^{-10}$ ft-lb/ft^2-sec-°R^4
σ = nose or leading edge radius of body or wing, ft
$\bar{\gamma}$ = ratio of specific heats
β = constant in planetary density-altitude relation $\rho = \rho_0 e^{-\beta y}$. Here ρ_0 = reference density, slugs/ft^3, β = const, and y = altitude; $\rho_0 = 0.0027$, $(1/\beta) = 23{,}500$ for earth (for other planets, see Ref. 3)

Subscripts

f = condition at end of power boost or condition at beginning of unpowered glide
l = local conditions
r = recovery conditions
s = stagnation conditions
sl = sea level conditions
av = average values
i = initial conditions
max = maximum values
w = wall conditions

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I. Introduction

THE fundamental equations of entry are

$$L = mg\cos\theta - mV^2(\cos\theta/R) - mV^2(d\theta/ds) \tag{1}$$

$$-D = -mg\sin\theta + (m/2)(dV^2/ds) \tag{2}$$

which may be combined to give the following equation:

$$\frac{dV^2}{ds} - \frac{2}{(L/D)}V^2\left(\frac{d\theta}{ds}\right) + \frac{2g\cos\theta}{(L/D)}\left(1 - \frac{V^2}{gR}\right) - 2g\sin\theta = 0 \tag{3}$$

For constant lift-drag ratio entry, exact analytical solutions are impossible. However, several first-order approximate solutions are available. In the case of ballistic entry without lift at large angles of inclination, Gazley,[1] Allen and Eggers,[3] and Chapman[2] obtained their first-order approximations by neglecting both the gravity force term and the centrifugal force term in the fundamental equations (1) and (2). In the case of gliding entry at small angles of inclination and positive lift-drag ratio, Allen and Eggers[3] obtained their first-order approximations by neglecting both the second term and the fourth term in the basic equation (3). In the case of gliding entry at large angles of inclination and negative lift-drag ratio, Loh obtained his first-order approximations by neglecting the third term of the basic equation (3). Because of the terms being neglected in the first-order theories, the solutions obtained therein are limited in a relatively narrow region of entry applications. Only recently, a second-order solution[9] and a unified solution of entry mechanics were developed by Loh. Although these solutions cover both glide and ballistic entry at either small or large angles of inclination, they are limited to constant lift-drag ratio entries. It is the purpose of this paper to present a few exact solutions for variable lift-drag ratio entries.

Constant lift-drag ratio trajectories were discussed by the forementioned authors. Constant lift-drag ratio requires variable angles of inclination (either large or small). As discussed in the previous papers,[5, 9, 10] large angles of inclination result in a higher rate of heat transfer but a lower amount of total heat input to the vehicle, whereas small angles of inclination result in a lower rate of heat transfer but a higher amount of total heat input to the vehicle. Both rate of heat transfer and amount of total heat input to the vehicle might raise the vehicle temperature above tolerable limit during certain phases of planetary atmosphere re-entry. Since the critical portion of atmosphere re-entry usually occurs either at relatively lower atmosphere or relatively shorter length of time, variable lift-drag ratio flight, although difficult, is not impossible. Trajectories under variable lift-drag ratio may be controlled in such a way that the peak deceleration, heating, etc., experienced by the vehicle are held to a tolerable amount within a short length of time or within a certain phase of lower atmosphere penetration. Control of vehicle trajectories in such a way may be made for 1) approximate constant deceleration flight or constant aerodynamic load factor flight, 2) constant rate of average heat input flight or constant average radiation equilibrium temperature flight, 3) constant rate of stagnation point input flight or constant stagnation region radiation equilibrium temperature flight, and 4) constant angle of inclination flight. Two kinds of exact analytical solutions will be discussed; they are as follows:

1) Exact analytical solutions of practical interest. These analytical solutions take the form ρV^n = const or angle of inclination is constant, where items 1–4 in the foregoing belong in this category.

2) Exact analytical solutions of general interest. These analytical solutions hold drag coefficient constant and express the angle of inclination as a function of density and an integer, i.e., $\sin\theta = f[(a/n)\rho]$.

In the case of exact analytical solutions of practical interest, if the appropriate minor terms are disregarded or simplified to those specified in the approximate solutions,[5] the present exact solutions also are reduced precisely to those solutions given previously.[5] The exact solutions developed here, such as the approximate constant deceleration entry, are especially useful for proper entry into Jupiter, where entry deceleration is most critical.

II. Analysis

In a nonrotating two-dimensional inertial coordinate sytem with its origin at the center of the earth or planet and the gravitational field g during the entry portion assumed as constant, the equations of motion in the directions normal and tangential to the trajectory are

$$L - mg\cos\theta = -mV^2(\cos\theta/R) - mV^2(d\theta/ds) \tag{4}$$

$$-D + mg\sin\theta = (m/2)(dV^2/ds) = m(dV/dt) \tag{5}$$

From the usual approximate exponential planetary atmosphere,† [2–4]

$$\rho = \rho_0 e^{-\beta y} \tag{6}$$

and

$$\beta = -(1/\rho)(d\rho/dy) = \bar{M}g/R'T \tag{7}$$

It is to be noted[3] here that the molecular weight $\bar{M}$, gas constant R', and temperature T of the planet's atmosphere enter the equation here only in the parameter β, which represents the local density gradient in the planetary atmosphere. Putting

$$L = \tfrac{1}{2} C_L \rho V^2 A$$

$$D = \tfrac{1}{2} C_D \rho V^2 A$$

using the kinematic relation,

$$-\sin\theta = dy/ds$$

into Eqs. (4) and (5) and rearranging terms, one obtains

$$\frac{d\cos\theta}{d(\beta y)} - \left(\frac{1}{\beta R_0}\right)\cos\theta\left(\frac{gR_0}{V^2} - 1\right) = -\frac{1}{2}\left(\frac{L}{D}\right) \times \left(\frac{C_DA}{m\beta}\right)\rho \tag{8}$$

$$\frac{d(V^2/gR_0)}{d(\beta y)} - \left(\frac{C_DA}{m\beta}\right)\rho\frac{(V^2/gR_0)}{\sin\theta} = -\left(\frac{2}{\beta R_0}\right) \tag{9}$$

substituting Eq. (6) into (8) and (9) and simplifying, one obtains

$$\frac{d\cos\theta}{d\rho} + \left(\frac{1}{\beta R_0}\right)\frac{\cos\theta}{\rho}\left(\frac{gR_0}{V^2} - 1\right) = \frac{1}{2}\left(\frac{L}{D}\right)\left(\frac{C_DA}{m\beta}\right) \tag{10}$$

$$\frac{d(V^2/gR_0)}{d\rho} + \left(\frac{C_DA}{m\beta}\right)\frac{(V^2/gR_0)}{\sin\theta} = \left(\frac{2}{\beta R_0}\right)\frac{1}{\rho} \tag{11}$$

Equations (10) and (11) are the exact equations of motion of entry into a planetary atmosphere. Exact solutions of practical interest will be presented first, and exact solutions of general interest will be given in the next section.

A. Exact Analytical Solutions of Practical Interest

1. Entry at constant ρV^n

Many entries of major interest belong to this category, for example, 1) constant Reynolds number entry $n = 1$ (approximate sense), 2) constant deceleration entry (approximate sense), 3) constant aerodynamic load factor entry ($n = 2$), 4) constant rate of heat input entry $n = 2$ (approximate sense), and 5) constant equilibrium skin temperature entry $n = 3$ (approximate sense). Detailed derivations of these entries for altitude, θ, (dV/dt), s, R, t, L/D required, will be given later.

Entry at constant value of ρV^n dictates that

$$\rho V^n = \rho_i V_i^n = \text{const} = k_1 \tag{12}$$

Here n and k_1 are constants, and subscript i indicates the initial condition at the beginning of variable lift-drag entry. Differentiating Eq. (3), one obtains

$$d\rho = -(n/2)k_1(dV^2/V^{n+2}) \tag{13}$$

It is always desirable to derive solutions of all the unknowns in terms of one independent variable, say V.

a. Altitude y. From Eqs. (6) and (12), one obtains

$$y = (1/\beta)\ln(\rho_0 V^n/k_1) \tag{14}$$

b. Angle of inclination θ. Substituting Eqs. (12) and (13) into (11) and simplifying, one obtains

$$\sin\theta = \frac{k_1(C_DA/m\beta)}{(2/n)V^n + (2g/\beta)V^{n-2}} \tag{15}$$

c. Deceleration (dV/dt) and maximum deceleration $(dV/dt)_{max}$. Substituting Eqs. (12) and (15) into (5), one obtains

$$\frac{dV}{dt} = -\frac{1}{2}\left(\frac{C_DA}{m}\right)\frac{k_1}{V^{n-2}} + g\frac{k_1(C_DA/m\beta)}{(2/n)V^n + (2g/\beta)V^{n-2}} \tag{16}$$

Maximum deceleration occurs at

$$(d/dV)(dV/dt) = 0 \tag{17}$$

This gives

$$\left(\frac{n}{2} - 1\right)\left(\frac{\beta}{2g}\right)\left[\left(\frac{2}{n}\right)V^2 + \left(\frac{2g}{\beta}\right)\right]^2 - \left(\frac{n}{2}\right)\left[\left(\frac{2}{n}\right)V^2 + \left(\frac{2g}{\beta}\right)\right] + \left(\frac{2g}{\beta}\right) = 0 \tag{18}$$

† This density relationship is based on the assumption of an isothermal gas in a uniform gravitational field.[3]

and its solution is

$$\left(\frac{2}{n}\right) V^2 + \left(\frac{2g}{\beta}\right) = \frac{+(n/2) \pm \{(n/2)^2 - 4[(n/2) - 1]\}^{1/2}}{[(n/2) - 1](\beta/g)} \quad (19)$$

Equation (19) indicates that maximum deceleration does not exist for $n \geq 2$, which are the cases of major interest here. The deceleration increases continuously as the velocity decreases and approaches an asymptote. (When $n = 2$, it is very nearly a constant deceleration entry.)

d. Flight distance along flight path s. From Eq. (16), one obtains

$$\frac{dV}{dt} = V\frac{dV}{ds} = \frac{1}{2}\left(\frac{C_DA}{m}\right)\frac{k_1}{V^{n-2}} + g\frac{k_1(C_DA/m\beta)}{(2/n)V^n + (2g/\beta)V^{n-2}}$$

or

$$ds = -\frac{(2/n)}{(C_DA/m)k_1}dV^n - \frac{(2g/\beta)[n/(n-2)]}{(C_DA/m)k_1}dV^{n-2} \quad (20)$$

Equation (20) may be integrated to give

$$s - s_i = \frac{(2/n)}{(C_DA/m)k_1}(V_i^n - V^n) + \frac{(2g/\beta)[n/(n-2)]}{(C_DA/m)k_1}(V_i^{n-2} - V^{n-2}) \quad (21)$$

e. Range R. One obtains from the geometrical relationship

$$dR = \cos\theta ds = (1 - \sin^2\theta)^{1/2}ds \quad (22)$$

Substituting Eq. (15) into (22), one obtains

$$R = \frac{2}{(C_DA/m)k_1} \times \int_V^{V_i}\left\{1 - \left[\frac{k_1(C_DA/m)}{(2/n)V^n + (2g/\beta)V^{n-2}}\right]^2\right\}^{1/2}\left[\left(\frac{1}{n}\right)dV^n + \left(\frac{g}{\beta}\right)\left(\frac{n}{n-2}\right)dV^{n-2}\right] \quad (23)$$

When $n = 2$, Eq. (23) reduces to

$$R = \frac{1}{(C_DA/m)k_1} \times \int_V^{V_i}\frac{\{[V^2 + (2g/\beta)]^2 - [k_1(C_DA/m)]^2\}^{1/2}}{[V^2 + (2g/\beta)]}d\left[V^2 + \left(\frac{2g}{\beta}\right)\right] \quad (24)$$

which yields the solution

$$R = \frac{1}{(C_DA/m)k_1}\left\{\left[\left(V_i^2 + \frac{2g}{\beta}\right)^2 - \left(k_1\frac{C_DA}{m}\right)^2\right]^{1/2} - \left[\left(V^2 + \frac{2g}{\beta}\right)^2 - \left(k_1\frac{C_DA}{m}\right)^2\right]^{1/2} + k_1\left(\frac{C_DA}{m}\right)\sin^{-1} \times \left[\frac{k_1(C_DA/m)}{V_i^2 + (2g/\beta)}\right] - k_1\left(\frac{C_DA}{m}\right)\sin^{-1}\left[\frac{k_1(C_DA/m)}{V^2 + (2g/\beta)}\right]\right\} \quad (25)$$

When $n \neq 2$, Eq. (23) easily can be integrated graphically

f. Time of flight t. Using Eq. (20), one obtains

$$t = \int\frac{ds}{V} = \frac{[2/(n-1)]}{(C_DA/m)k_1}(V_i^{n-1} - V^{n-1}) + \frac{(2g/\beta)[n/(n-3)]}{(C_DA/m)k_1}(V_i^{n-3} - V^{n-3}) \quad (26)$$

g. Required lift-drag ratio L/D. Equation (10) may be rewritten as

$$\left(\frac{L}{D}\right) = \frac{2}{(C_DA/m\beta)}\left[-\sin\theta\left(\frac{d\theta}{d\rho}\right) + \left(\frac{1}{\beta R_0}\right)\frac{\cos\theta}{\rho} \times \left(\frac{gR_0}{V^2} - 1\right)\right] \quad (27)$$

Differentiating equation (15), one obtains

$$\cos\theta\left(\frac{d\theta}{d\rho}\right) = \frac{(C_DA/m\beta)(2/n)V^{2n} + (2g/\beta)[(n-2)/n](C_DA/m\beta)V^{2(n-1)}}{(2/n)V^n + (2g/\beta)V^{n-2}} \quad (28)$$

Substituting Eqs. (12, 15, and 28) into (27), one obtains

$$\left(\frac{L}{D}\right) = \left(\frac{2}{\beta R_0}\right)\frac{1}{(C_DA/m\beta)}\frac{(1 - \{k_1(C_DA/m\beta)/[(2/n)V^n + (2g/\beta)V^{n-2}]\}^2)^{1/2}}{(k_1/V^n)}\left(\frac{gR_0}{V^2} - 1\right) - \left\{\left[\frac{(2/n)V^n + (2g/\beta)V^{n-2}}{k_1(C_DA/m\beta)}\right]^2 - 1\right\}^{1/2}\left[\frac{(4/n)V^{2n} + (4g/\beta)[(n-2)/n]V^{2(n-1)}}{(2/n)V^n + (2g/\beta)V^{(n-2)}}\right] \quad (29)$$

It should be noticed that when the minor term $(2g/\beta)$ of Eq. (11) and consequently all terms containing the $(2g/\beta)$ term in all the solutions obtained in this paper are neglected, the exact solutions (15, 16, 21, 23, 25, 26, and 29) all are reduced to the corresponding approximate solutions published previously.[5] It is to be noticed here that the error introduced [for example, Eq. (15)] by the approximate solution published previously[5] is in the order of

$$\frac{\text{approximate solution}}{\text{exact solution}} \cong 1 + \left[\frac{(2g/\beta)}{V^2}\right]$$

This shows that the error is negligibly small (less than 1% when V is in the neighborhood of orbital speed or greater) when V is large, but the error is increased when V is reduced during lower atmosphere penetration. At least six entries belong to the category $\rho V^n = k_1$.

a. Constant deceleration entry (approximate sense). Equation (5) reads

$$dV/dt = -(C_DA/m)\tfrac{1}{2}\rho V^2 + g\sin\theta \cong -(C_DA/m)\tfrac{1}{2}\rho V^2 \quad (5a)$$

One sees immediately that, for a constant deceleration entry at small angle of inclination θ, ρV^2 is a constant. Therefore, when $n = 2$, the solutions obtained here become the solutions for entry at nearly constant deceleration.

b. Constant aerodynamic load factor entry. The aerodynamic load factor is $(\frac{1}{2})\rho V^2$. One sees immediately that, for a constant aerodynamic load factor entry, $n = 2$.

c. Constant time rate of heat input entry. The time rate of average heat input given in Refs. 3 and 4 is

$$dH_{av}/dt = (1/S)(dQ/dt) = \tfrac{1}{4}C_F'\rho V^3$$

One sees immediately that, for a constant time rate of average heat input entry, $n = 3$.

The time rate of stagnation heat input‡ given in Refs. 3 and 4 is

$$dH_s/dt = K'(\rho/\sigma)^{1/2}V^3$$

One sees immediately that, for a constant time rate of stagnation heat input entry, $n = 6$. This is because $\rho^{1/2}V^3 =$ const and $(\rho^{1/2}V^3)^2 = \rho V^6 =$ const.

d. Constant radiation equilibrium skin temperature entry. This entry is the same as the constant time rate of heat input entry. The reason is that, for a given surface temperature, rate of heat radiated out is equal to $\epsilon k' T_w^4$. Using this quantity and the rate of heat input expression given in Refs. 3–5, a velocity-density or velocity-altitude relationship can be determined such that the heat input to the body is exactly equal to the heat radiated out. However, it is necessary to exert lift, aerodynamic, or even reaction when necessary to traverse such trajectory along the specified path, as is absolutely required for all the variable lift-drag ratio flights presented in this paper.

e. Other entry solutions. Other entry solutions that are desired to maintain a $\rho V^n =$ const also may be obtained from the solutions presented here. For example, for a constant Reynold number[10] flight (when the viscosity μ is treated as a constant), $n = 1$.

$$\left(\frac{dV}{dt}\right) = -\frac{1}{2}\left(\frac{C_DA}{m}\right)\rho V^2 + g\sin\theta =$$

$$-\frac{1}{2}\left(\frac{C_DA}{m}\right)\rho\left(\frac{2g}{\beta}\right)e^{-(C_DA/m\beta\sin\theta)\rho}\left\{\frac{V_i^2}{e^{-(C_DA/m\beta\sin\theta)\rho}} + \left(\frac{2g}{\beta}\right)\ln\left(\frac{\rho}{\rho_i}\right) + \left(\frac{2g}{\beta}\right)\times \sum_{n=1}^{\infty}\left[\frac{(C_DA/m\beta\sin\theta)^n(\rho^n - \rho_i^n)}{n(n!)}\right]\right\} + g\sin\theta \quad (33)$$

c. Distance along flight path s. From $\rho = \rho_0 e^{-\beta y}$ and geometrical relationship $(dy/ds) = -\sin\theta$, one obtains

$$d\rho = -\rho\beta\, dy = \beta\rho\sin\theta\, ds \quad (34)$$

Therefore

$$(s - s_i) = \int_{\rho i}^{\rho}\frac{d\rho}{\beta\sin\theta\rho} = \frac{1}{\beta\sin\theta}\ln\left(\frac{\rho}{\rho_i}\right) \quad (35)$$

d. Range R.

$$R = \int\cos\theta ds = \cos\theta(s - s_i)$$
$$= (1/\beta)\cot\theta\ln(\rho/\rho_i) \quad (36)$$

e. Time of flight t.

$$t = \int\frac{ds}{V} = \int_{\rho i}^{\rho}\frac{d\rho}{\beta\sin\theta\rho V}$$

$$= \frac{1}{\beta\sin\theta}\int_{\rho i}^{\rho}\frac{e^{(C_DA/m\beta\sin\theta)(\rho/2)}\,d\rho}{\rho\left\{[V_i^2/e^{-(C_DA/m\beta\sin\theta)\rho i}] + (2g/\beta)\ln(\rho/\rho_i) + (2g/\beta)\sum_{n=1}^{\infty}[(C_DA/m\beta\sin\theta)^n/n(n!)](\rho^n - \rho_i^n)^{1/2}\right\}} \quad (37)$$

Equation (37) may be integrated graphically.

f. Required lift-drag ratio L/D. When $\theta =$ const, Eqs. (27) and (32) give

$$\frac{L}{D} = \left(\frac{2}{\beta R_0}\right)\frac{\cos\theta}{(C_DA/m\beta)}\left(\frac{1}{\rho}\right)\left\{\frac{gR_0e^{(C_DA/m\beta\sin\theta)\rho}}{V_i^2e^{(C_DA/m\beta\sin\theta)\rho i} + (2g/\beta)\left\{\ln(\rho/\rho_i) + \sum_{n=1}^{\infty}[(C_DA/m\beta\sin\theta)^n/n(n!)](\rho^n - \rho_i^n)\right\}} - 1\right\} \quad (38)$$

2. *Entry at constant angle of inclination*

Entry at constant angle of inclination dictates that

$$\theta = \theta_i = \text{const} = k_3 \quad (30)$$

In this case, various solutions of unknowns may be expressed most easily in terms of the independent variable ρ.

a. Velocity V. When angle θ is a constant, Eq. (11) readily may be solved. The solution is

$$V^2 = \frac{2g}{\beta}e^{-(C_DA/m\beta\sin\theta)\rho}\left[\int e^{(C_DA/m\beta\sin\theta)\rho}\frac{d\rho}{\rho} + C\right] \quad (31)$$

Since

$$e^x = 1 + (x^n/n!)$$

$$e^{(C_DA/m\beta\sin\theta)\rho} = 1 + \frac{(C_DA/m\ \beta\ \sin\theta)^n\ \rho^n}{n!}$$

Equation (31) may be integrated readily

$$V^2 = e^{-(C_DA/m\beta\sin\theta)\rho}\left\{\frac{V_i^2}{e^{-(C_DA/m\beta\sin\theta)\rho i}} + \left(\frac{2g}{\beta}\right)\times \ln\left(\frac{\rho}{\rho_i}\right) + \left(\frac{2g}{\beta}\right)\sum_{n=1}^{\infty}\left[\frac{(C_DA/m\beta\sin\theta)^n(\rho^n - \rho_i^n)}{n(n!)}\right]\right\} \quad (32)$$

b. Deceleration dV/dt. From Eq. (5), one obtains

‡ Viscosity μ was assumed inversely proportional to square root of temperature T.

It should be noticed that, when the minor terms containing $(2g/\beta)$ are neglected, Eqs. (32, 33, and 35–38) all are reduced to the corresponding approximate solutions published previously.[5]

Note that Eq. (38) indicates that, when θ is large, (L/D) becomes small, and, therefore, for either ballistic entry (where $L/D = 0$) or small (L/D) entry at large angles of inclination, the trajectory is very close to a constant angle of inclination entry trajectory.

g. Special case where $(L/D) = 0$. When $(L/D) = 0$ and the trajectory becomes ballistic entry at constant angles of inclination, the present solutions, when minor terms are neglected, are reduced precisely to those solutions published previously for ballistic entry at large and constant angles of inclination.[1–3, 5]

B. Exact Analytical Solutions of General Interest§

The following cases of entry at constant C_D and variable (L/D) will be discussed:

$$\sin\theta = (a/n)\rho$$
$$\sin\theta = a\rho\ln\rho$$
$$\sin\theta = (a/n)\rho^{1-n}$$
$$1/\sin\theta = [1/(a/n)\rho] + (1/a\rho\ln\rho)$$
$$1/\sin\theta = [1/(a/n)\rho] + (1/a\rho\ln\rho) + [1/(a/n)\rho^{1-n}]$$

The basic equations (10) and (11) become

$$\frac{dx}{d\rho} + a\frac{x}{\sin\theta} = \frac{2b}{\rho} \quad (39)$$

§ Practical applications of those solutions currently are not visualized.

$$\frac{d\cos\theta}{d\rho} + b\,\frac{\cos\theta}{\rho}\left(\frac{1}{x} - 1\right) = \left(\frac{a}{2}\right)\left(\frac{L}{D}\right) \tag{40}$$

Here

$$a = (C_DA/m\beta) = \text{const}$$
$$b = (1/\beta R_0) = \text{const} \tag{41}$$
$$x = (V^2/gR_0) = \text{nondimensional variable}$$

1. Entry at $\sin\theta = (a/n)\rho$

Here n is a constant that could be any integer or fraction of an integer, $-\infty < n < +\infty$.

a. Angle of inclination θ. The specified entry condition is

$$\sin\theta = (a/n)\rho \tag{42}$$

b. Altitude density ρ and altitude y. Substituting Eq. (42) into (39), one obtains

$$(dx/d\rho) + n(x/\rho) = 2b/\rho \tag{43}$$

The solution of Eq. (43) is

$$x = (2b/n) + (\rho_i/\rho)^n[x_i - (2b/n)] \tag{44}$$

or

$$\rho = \rho_i\left[\frac{x_i - (2b/n)}{x - (2b/n)}\right]^{1/n} \tag{45}$$

$$y = -\frac{1}{\beta}\ln\left\{\left(\frac{\rho_i}{\rho_0}\right)\left[\frac{x_i - (2b/n)}{x - (2b/n)}\right]^{1/n}\right\} \tag{46}$$

c. Deceleration dV/dt and maximum deceleration $(dV/dt)_{\max}$. Substituting Eqs. (42) and (44) into (5), one obtains

$$\left(\frac{dV}{dt}\right) = -\left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)\rho\left[\left(\frac{2b}{n}\right) + \left(\frac{\rho_i}{\rho}\right)^n \times \left(x_i - \frac{2b}{n}\right)\right] + g\left(\frac{a}{n}\right)\rho \tag{47}$$

Maximum deceleration occurs at

$$(d/d\rho)(dV/dt) = 0$$

This gives

$$\left(\frac{\rho^*}{\rho_i}\right) = \left[\frac{(1-n)[x_i - (2b/n)]}{(2/R_0)(a/n)(m/C_DA) - (2b/n)}\right]^{1/n} \tag{48}$$

Here ρ^* is the altitude density at which $(dV/dt)_{\max}$ occurs.

Substituting Eq. (48) into (47), one obtains the maximum deceleration in terms of x_i, (m/C_DA), and n, $(2b/n)$, (a/n), $(2/R_0)$, and ρ_i.

d. Distance along flight path s. Substituting Eq. (42) into (34), one obtains

$$s - s_i = \int_{\rho_i}^{\rho}\frac{d\rho}{\beta\rho(a/n)\rho} = \left(\frac{n}{a\beta}\right)\left(\frac{1}{\rho_i} - \frac{1}{\rho}\right) \tag{49}$$

e. Range R.

$$dR = \cos\theta ds = \left(\frac{n}{a\beta}\right)\int_{\rho_i}^{\rho}\frac{(1-\sin^2\theta)^{1/2}d\rho}{\rho^2} \tag{50}$$

Substituting Eq. (42) into (50), one obtains

$$R = \left(\frac{1}{\beta}\right)\left[\frac{\{1 - [(a/n)\rho_i]^2\}^{1/2}}{[(a/n)\rho_i]} - \frac{\{1 - [(a/n)\rho]^2\}^{1/2}}{[(a/n)\rho]} + \sin^{-1}\left(\frac{a}{n}\rho_i\right) - \sin^{-1}\left(\frac{a}{n}\rho\right)\right] \tag{51}$$

f. Time of flight t.

$$t = \int\frac{ds}{V} = \int_{\rho i}^{\rho}\left(\frac{n}{a\beta}\right)\frac{d\rho}{\rho^2V} \tag{52}$$

Substituting Eq. (44) into (52), one obtains

$$t = \left(\frac{n}{a\beta}\right)\left[\frac{1}{(gR_0)^{1/2}}\right] \times \int_{\rho i}^{\rho}\frac{d\rho}{\{(2b/n)\rho^4 + \rho^4(\rho_i/\rho)^n[x_i - (2b/n)]\}^{1/2}} \tag{53}$$

g. Required (L/D). From Eq. (40), one obtains

$$\left(\frac{L}{D}\right) = \left(\frac{2}{a}\right)\left[b\,\frac{(1-\sin^2\theta)^{1/2}}{\rho} \times \left(\frac{1}{x} - 1\right) - \sin\theta\left(\frac{d\theta}{d\rho}\right)\right] \tag{54}$$

Differentiating Eq. (42) with respect to ρ, one obtains

$$\cos\theta(d\theta/d\rho) = (a/n) \tag{55}$$

Substituting Eqs. (42, 44, and 55) into (54), one obtains (L/D) in terms of ρ only

$$\left(\frac{L}{D}\right) = \left(\frac{2}{a}\right)\left\{b\,\frac{[1 - (a/n)^2\rho^2]^{1/2}}{\rho} \times \left[\frac{1}{(2b/n) + (\rho_i/\rho)^n[x_i - (2b/n)]} - 1\right] - \left(\frac{a}{n}\right)^2 \times \rho\left(\frac{1}{[1 - (a/n)^2\rho^2]^{1/2}}\right)\right\} \tag{56}$$

2. Entry at $\sin\theta = a\rho\ln\rho$

a. Angle of inclination θ. The angle of inclination θ vs altitude density ρ again is given as a preselected entry path. This entry path again has to be fulfilled by the required variable (L/D) to be determined later in order to satisfy Eq. (40):

$$\sin\theta = a\rho\ln\rho \tag{57}$$

b. Altitude density ρ. Substituting Eq. (57) into (39), one obtains

$$(dx/d\ln\rho) + (x/\ln\rho) = 2b \tag{58}$$

Equation (58) readily may be solved

$$x = e^{-\int(d\ln\rho/\ln\rho)}\left(2b\int e^{\int(d\ln\rho/\ln\rho)}\,d\ln\rho + C\right) \tag{59}$$

$$= b\ln\rho + (\ln\rho_i/\ln\rho)(x_i - b\ln\rho_i) \tag{60}$$

c. Deceleration (dV/dt) and maximum deceleration $(dV/dt)_{\max}$.

$$\left(\frac{dV}{dt}\right) = -\left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)\rho\left[b\ln\rho + \frac{\ln\rho_i}{\ln\rho}(x_i - b\ln\rho_i)\right] + ga\rho\ln\rho \tag{61}$$

The maximum deceleration occurs at

$$(d/d\rho)(dV/dt) = 0$$

The density ρ^* at which the maximum deceleration occurs is

$$\left[ga - \left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)b\right](1 + \ln\rho) = \left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)\ln\rho_i(x_i - b\ln\rho_i)\left(\frac{1}{\ln\rho}\right)\left[1 - \left(\frac{1}{\ln\rho}\right)\right] \tag{62}$$

Let

$$[ga - (gR_0/2)(C_DA/m)b] = n_1 \qquad (63)$$

$$(gR_0/2)(C_DA/m) \ln\rho_i(x_i - b \ln\rho_i) = n_2 \qquad (64)$$

One obtains

$$n_1(1 + \ln\rho) = n_2\left[\frac{1}{\ln\rho} - \frac{1}{(\ln\rho)^2}\right] \qquad (65)$$

$$n_1(\ln\rho)^3 + n_1(\ln\rho)^2 - n_2(\ln\rho) + n_2 = 0 \qquad (66)$$

The solution of Eq. (66) may be obtained readily by the ordinary method of solving algebraic equations of the third degree.

d. Required (L/D). Differentiating Eq. (57) with respect to ρ, one obtains

$$\cos\theta(d\theta/d\rho) = a[\rho(d\ln\rho/d\rho) + \ln\rho] = a(1 + \ln\rho) \qquad (67)$$

Substituting Eqs. (57, 60, and 67) into (54), one obtains

$$\left(\frac{L}{D}\right) = \left(\frac{2}{a}\right)\left\{b\,\frac{[1 - (a\rho\ln\rho)^2]^{1/2}}{\rho} \times \left[\frac{1}{b\ln\rho + (\ln\rho_i/\ln\rho)(x_i - b\ln\rho_i)} - 1\right] - (a^2\rho\ln\rho)\left(\frac{1 + \ln\rho}{[1 - (a\rho\ln\rho)^2]^{1/2}}\right)\right\} \qquad (68)$$

3. Entry at sin

$$\theta = (a/n)\rho^{1-n}$$

a. Angle of inclination θ. The angle of inclination vs altitude density ρ again is preselected as

$$\sin\theta = (a/n)\rho^{1-n} \qquad (69)$$

b. Altitude density ρ. Substituting Eq. (69) into (40), one obtains

$$(dx/d\rho) + (nx/\rho^{1-n}) = 2b/\rho \qquad (70)$$

$$x = e^{-\int(n/\rho^{1-n})d\rho}\left(2b \int e^{\int(n/\rho^{1-n})d\rho}\,\frac{1}{\rho}\,d\rho + C\right) = \left(\frac{1}{e^{\rho^n}}\right)\left[\left(\frac{2b}{n}\right)\int e^{\rho^n}\,d\ln\rho^n + C\right] \qquad (71)$$

let

$$e^{\rho^n} = 1 + \sum_{p=1}^{\infty}\frac{(\rho^n)^p}{p!} \qquad (72)$$

Substituting Eq. (72) into (71) and performing integration, one obtains

$$x = \frac{1}{e^{\rho^n}}\left[(2b)\ln\left(\frac{\rho}{\rho_i}\right) + \left(\frac{2b}{n}\right)\sum_{p=1}^{\infty}\frac{(\rho^n)^p - (\rho_i^n)^p}{p(p!)} + x_ie^{\rho_i^n}\right] \qquad (73)$$

c. Deceleration (dV/dt). Substituting Eqs. (69) and (73) into (3), one obtains

$$\frac{dV}{dt} = -\left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)\rho e^{-\rho^n}\left[2b\ln\left(\frac{\rho}{\rho_i}\right) + \frac{2b}{n}\sum_{p=1}^{\infty}\frac{(\rho^n)^p - (\rho_i^n)^p}{p\cdot p!} + x_ie^{\rho_i^n}\right] + g\left(\frac{a}{n}\right)\rho^{1-n} \qquad (74)$$

d. Required (L/D). Differentiating Eq. (69) with respect to ρ, one obtains

$$\cos\theta(d\theta/d\rho) = (a/n)(1 - n)\rho^{-n} \qquad (75)$$

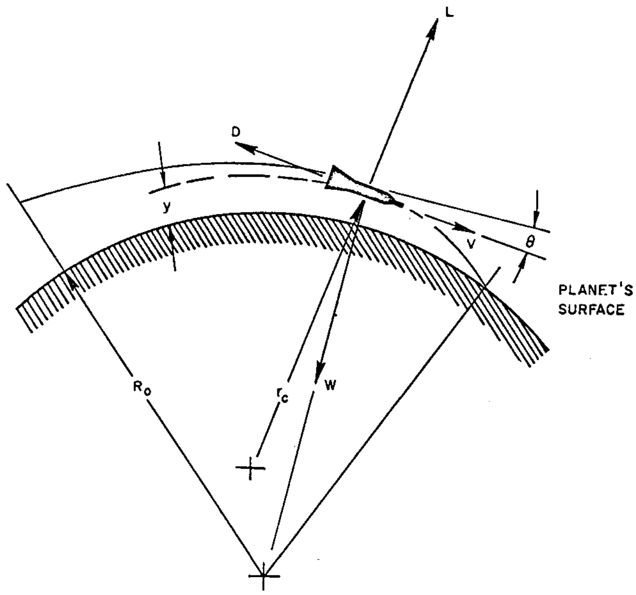

Fig. 1 Two-dimensional flight

Substituting Eqs. (69), (70), and (75) into (54), one obtains

$$\left(\frac{L}{D}\right) = \left(\frac{2}{a}\right)\left[b\,\frac{\{1 - [(a/n)\rho^{1-n}]^2\}^{1/2}}{\rho}\left(\frac{1}{x} - 1\right) - \frac{(a/n)^2\rho^{1-2n}(1 - n)}{\{1 - [(a/n)\rho^{1-n}]^2\}^{1/2}}\right] \qquad (76)$$

4. Entry at $(1/\sin\theta) = [1/(a/n)\rho] + (1/a\rho\ln\rho)$

It should be noticed that this case is one whose reciprocal is equal to the sum of the reciprocal of the function described in cases 1 and 2 here.

a. Angles of inclination θ. The angle of inclination vs altitude density ρ again is preselected as

$$1/\sin\theta = [1/(a/n)\rho] + (1/a\rho\ln\rho) \qquad (77)$$

b. Altitude density ρ. Substituting Eq. (77) into (39), one obtains

$$\rho(dx/d\rho) + [n + (1/\ln\rho)]x = 2b \qquad (78)$$

The solution of Eq. (78) is

$$x = \exp\left[\int -\left(n + \frac{1}{\ln\rho}\right)d\ln\rho\right]\left\{2b\int \exp \times \left[\int\left(n + \frac{1}{\ln\rho}\right)d\ln\rho\right]d\ln\rho + C\right\} = \rho^{-n}\,\frac{1}{\ln\rho}\left(2b\int \rho^n\ln\rho\,d\ln\rho + C\right) \qquad (79)$$

Let

$$\ln\rho^n = Z \qquad \rho^n = e^z \qquad (80)$$

Therefore, the integral becomes

$$\int \rho^n\ln\rho\,d\ln\rho = \frac{1}{n^2}\int Ze^zdZ = \left(\frac{1}{n^2}\right)(Ze^z - e^z) = \frac{1}{n^2}\,e^z(Z - 1) \qquad (81)$$

Substituting Eq. (81) into (79) and determining the constant by initial conditions, one obtains

$$x = (1/\rho^n\ln\rho)\{(2b/n^2)[(n\ln\rho - 1)\rho^n - (n\ln\rho_i - 1)\rho_i^n] + x_i\rho_i^n\ln\rho_i\} \qquad (82)$$

c. Deceleration (dV/dt). Substituting Eqs. (77) and (82) into (3), one obtains

$$\frac{dV}{dt} = -\left(\frac{gR_0}{2}\right)\left(\frac{C_DA}{m}\right)\rho\left\{x_i\rho_i{}^n \ln\rho_i + \frac{1}{\rho^n \ln\rho}\left[\left(\frac{2b}{n^2}\right)(n\ln\rho - 1)\rho^n - \left(\frac{2b}{n^2}\right)(n\ln\rho_i - 1)\rho_i{}^n\right]\right\} + g\left[\frac{1}{\{1/[(a/n) - \rho]\} + (1/a\rho\ln\rho)}\right] \quad (83)$$

d. Required (L/D). Differentiating Eq. (77) with respect to ρ and after simplification, one obtains

$$\cos\theta\left(\frac{d\theta}{d\rho}\right) = (1 - \sin^2\theta)^{1/2}\left(\frac{d\theta}{d\rho}\right) = \frac{a(1 + \ln\rho)}{n\ln\rho + 1} \quad (84)$$

Substituting Eqs. (77, 82, and 84) into (54), one obtains the complete expression of required (L/D) in terms of altitude density ρ only.

5. Entry at $(1/\sin\theta) = [1/(a/n)\rho] + (1/a\rho\ln\rho) + [1/(a/n)\rho^{1-n}]$

It should be noticed that this case is one whose reciprocal is equal to the sum of the reciprocals of entry described in cases 1, 2, and 3.

a. Angle of inclination θ. The angle of inclination vs altitude density ρ again is preselected as

$$\frac{1}{\sin\theta} = \frac{1}{(a/n)\rho} + \frac{1}{a\rho\ln\rho} + \frac{1}{(a/n)\rho^{1-n}} \quad (85)$$

b. Altitude density ρ.

$$\rho(dx/d\rho) + [n + (1/\ln\rho) + n\rho^n]x = 2b \quad (86)$$

The solution of Eq. (86) is

$$x = \exp\left[-\int\left(n + \frac{1}{\ln\rho} + n\rho^n\right)d\ln\rho\right] \times \left\{2b\int \exp\left[\int\left(n + \frac{1}{\ln\rho} + n\rho^n\right)d\ln\rho\right]d\ln\rho + C\right\} \quad (87)$$

$$x = \left[\frac{1}{\rho^n(\ln\rho)e^{\rho^n}}\right]\left[(2b)\int \rho^n(\ln\rho)e^{\rho^n}d\ln\rho + C\right] \quad (88)$$

Let

$$\ln\rho^n = Z \qquad \rho^n = e^z \quad (89)$$

Therefore the integral becomes

$$\int \rho^n(\ln\rho)e^{\rho^n}d\ln\rho = \left(\frac{1}{n^2}\right)\left[(Z - 1)e^z + \sum_{p=1}^{\infty}\left(\frac{1}{1+p}\right)^2 e^{z(1+p)}\right] \quad (90)$$

Substituting Eq. (90) into (88) and determining constant C by initial conditions, one obtains

$$x = \left[\frac{1}{\rho^n(\ln\rho)e^{\rho^n}}\right]\left\{\left(\frac{2b}{n^2}\right)[(n\ln\rho - 1)\rho^n - (n\ln\rho_i - 1)\rho_i{}^n] + \sum_{p=1}^{\infty}\left(\frac{1}{1+p}\right)^2[\rho^{n(1+p)} - \rho_i{}^{n(1+p)}] + x_i[\rho_i{}^n(\ln\rho_i)e^{\rho_i{}^n}]\right\} \quad (91)$$

c. Deceleration (dV/dt). Substituting Eqs. (85) and (91) into (3), one obtains the deceleration (dV/dt) in terms of altitude density ρ only.

d. Required (L/D). Differentiating Eq. (85) with respect to ρ and after simplification, one obtains

$$\cos\theta\left(\frac{d\theta}{d\rho}\right) = (1 - \sin^2\theta)^{1/2}\left(\frac{d\theta}{d\rho}\right) = \frac{a[n + (1/\ln\rho) + (1/\ln\rho)^2 - n(n-1)\rho^n]}{[n + (1/\ln\rho) + n\rho^n]^2} \quad (92)$$

Substituting Eqs. (85, 91, and 92) into (54), one obtains the complete expression of required (L/D) in terms of altitude density ρ only.

III. Conclusions

In general, variable lift-drag ratios may be obtained in the following three ways:

1) Varying the angle of attack of a lifting vehicle.

2) Varying the drag coefficient at a constant lift coefficient (deploying a variable area drag device while the lifting surface maintains a fixed lift coefficient).

3) Varying the lift coefficient at a constant drag coefficient, such as using a body, the drag coefficient of which remains essentially constant when its lift coefficient varies (bodies with high parasite drag coefficients which dominate induced drag coefficients), or such as simultaneously changing the angle of attack and deploying a variable-area drag device (drag chute). The interrelation of drag coefficient and lift-drag ratio[||] at hypersonic speeds has been discussed by Chapman.[2] However, for the present paper, it was of interest only at the case where drag coefficient [#] is a constant and lift-drag ratio is a variable that is achieved either by aerodynamic forces or even reaction forces. The interrelation of drag coefficient and lift-drag ratio as investigated by Chapman was neglected in this paper.

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|| In general, C_L and C_D usually are related. This, of course, excludes use of reaction force to exert lift and also may exclude case 3 mentioned in the foregoing (see Chapman[2]).

Results on variable drag coefficient are to be published separately.